ALGEBRAIC GEOMETRY–MID SEMESTRAL EXAM
B. MATH. III
TIME: 3 HOURS

All rings are assumed to be commutative with identity. All the questions carry equal marks. Maximum score is 50.

You are allowed to bring a single page of notes.

(1) (i) Let A be a ring such that $A[x]$ is a Noetherian ring. Is A Noetherian ?
(ii) Let A be a ring and let I be an ideal of A. If I is free as an A-module, then what are its possible ranks ?
(iii) Let C be the ring of continuous real valued functions on $[0,1]$. Is C Noetherian ?
(iv) Let $A \to B$ be a ring homomorphism and let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{p}$ is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.

(2) Let M be a free module over A with basis $\{e_1, \cdots, e_n\}$. Let $x = \sum_{i=1}^n a_i e_i$ be an element of M, where $a_i \in A$ for all i, such that there exists $b_i \in A, 1 \leq i \leq n$ for which we have $\sum_{i=1}^n a_i b_i = 1$. Show that there exists a basis for M which contains x as a member.

(3) (i) Let B be an A-algebra and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $\theta : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $\theta(N)$ is a direct summand of N_B.
(ii) Let B be a flat A-algebra. Suppose that for any non-zero A-module M, we have $M_B = B \otimes_A M \neq 0$. Then show that for every A-module M, the mapping $x \mapsto 1 \otimes x$ of M into M_B is injective.

(4) (i) Let I be a finitely generated ideal of A such that A/I is a Noetherian ring. Suppose that I is nilpotent as an ideal. Show that A is a Noetherian ring.
(ii) k be a field and let K be its algebraic closure. Show that there is a $1-1$ correspondence between k-algebra homomorphisms $f : k[x_1, \cdots, x_n] \to K$ and tuples $(a_1, \cdots, a_n) \in L^n$ where L is some finite algebraic extension of k (which depends on the map f).

(5) Let A be a Noetherian ring, B a finitely generated A-algebra, G a finite group of A-automorphisms of B, and B^G the set of all elements of B which are left fixed by every element of G. Show that B^G is a finitely generated A-algebra.

(6) Let $f : M \to N$ be a homomorphism of A-modules, where M and N are both Noetherian A-modules. Assume that for some prime ideal $\mathfrak{p}$ of A, $f_{\mathfrak{p}}$ is an isomorphism. Show that there exists some $s \in A - \mathfrak{p}$ such that f_s is an isomorphism.

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